# Furstenberg transformations on Cartesian products of infinite-dimensional tori

P. A. Cecchi[1] and R. Tiedra de Aldecoa[2*]

[1] *Departamento de Matemáticas, Universidad de Santiago de Chile, Av. Alameda Libertador Bernardo O'Higgins 3363, Estación Central, Santiago, Chile*

[2] *Facultad de Matemáticas, Pontificia Universidad Católica de Chile, Av. Vicuña Mackenna 4860, Santiago, Chile*

E-mails: *pcecchib@gmail.com, rtiedra@mat.puc.cl*

**Abstract**

We consider in this note Furstenberg transformations on Cartesian products of infinite-dimensional tori. Under some appropriate assumptions, we show that these transformations are uniquely ergodic with respect to the Haar measure and have countable Lebesgue spectrum in a suitable subspace. These results generalise to the infinite-dimensional setting previous results of H. Furstenberg, A. Iwanik, M. Lemańzyk, D. Rudolph and the second author in the one-dimensional setting. Our proofs rely on the use of commutator methods for unitary operators and Bruhat functions on the infinite-dimensional torus.

**2010 Mathematics Subject Classification:** 28D10, 37A30, 37C40, 58J51.

**Keywords:** Furstenberg transformations, infinite-dimensional torus, commutator methods.

## 1 Introduction

We consider in this note the generalisation of Furstenberg transformations on Cartesian products $\mathbb{T}^d$ ($d \ge 2$) of one-dimensional tori [8, Sec. 2] to the case of Cartesian products $(\mathbb{T}^\infty)^d$ of infinite-dimensional tori. Using recent results on commutator methods for unitary operators [14, 15], we show under a $C^1$ regularity assumption on the perturbations that these transformations are uniquely ergodic with respect to the Haar measure on $(\mathbb{T}^\infty)^d$ and are strongly mixing in the orthocomplement of functions depending only on variables in the first torus $\mathbb{T}^\infty$ (see Assumption 3.2 and Theorem 3.3). Under a slightly stronger regularity assumption ($C^1$ + Dini continuous derivative), we also show that these transformations have countable Lebesgue spectrum in that orthocomplement (see Theorem 3.4). These results generalise to the infinite-dimensional setting previous results of H. Furstenberg, A. Iwanik, M. Lemańzyk, D. Rudolph and the second author [8, 10, 15] in the one-dimensional setting.

Apart from commutator methods, our proofs rely on the use of Bruhat test functions [5] which provide a natural analog to the usual $C^\infty$-functions which do not exist in our infinite-dimensional setting (see Section 3 for details). We also mention that all the results of this note apply to Furstenberg transformations on Cartesian products $(\mathbb{T}^n)^d$ of finite-dimensional tori $\mathbb{T}^n$ of any dimension $n \ge 1$. One just has to consider the particular case where the functions defining the Furstenberg transformations on $(\mathbb{T}^\infty)^d$ depend only on a finite number of variables.

*Supported by the Chilean Fondecyt Grant 1130168 and by the Iniciativa Cientifica Milenio ICM RC120002 "Mathematical Physics" from the Chilean Ministry of Economy.

Here is a brief description of the content of the note. First, we recall in Section 2 the needed facts on commutators of unitary operators and regularity classes associated to them. Then, we define in Section 3 the Furstenberg transformations on $(\mathbb{T}^\infty)^d$, and prove Theorems 3.3 and 3.4 on the unique ergodicity, strong mixing property and countable Lebesgue spectrum of these transformations.

We refer to [9, 11, 12, 13] and references therein for other works building on Furstenberg transformations.

**Acknowledgements.** The authors thank M. Măntoiu and J. Rivera-Letelier for discussions which partly motivated this note.

## 2 Commutator methods for unitary operators

We recall in this section some facts borrowed from [7, 14] on commutator methods for unitary operators and regularity classes associated to them.

Let $\mathcal{H}$ be a Hilbert space with scalar product $\langle\cdot,\cdot\rangle$ antilinear in the first argument, denote by $\mathscr{B}(\mathcal{H})$ the set of bounded linear operators on $\mathcal{H}$, and write $\|\cdot\|$ both for the norm on $\mathcal{H}$ and the norm on $\mathscr{B}(\mathcal{H})$. Let $A$ be a self-adjoint operator in $\mathcal{H}$ with domain $\mathcal{D}(A)$, and take $S \in \mathscr{B}(\mathcal{H})$. For any $k \in \mathbb{N}$, we say that $S$ belongs to $C^k(A)$, with notation $S \in C^k(A)$, if the map

$$\mathbb{R} \ni t \mapsto \mathrm{e}^{-itA} S \, \mathrm{e}^{itA} \in \mathscr{B}(\mathcal{H}) \tag{2.1}$$

is strongly of class $C^k$. In the case $k = 1$, one has $S \in C^1(A)$ if and only if the quadratic form

$$\mathcal{D}(A) \ni \varphi \mapsto \langle \varphi, SA\varphi \rangle - \langle A\varphi, S\varphi \rangle \in \mathbb{C}$$

is continuous for the topology induced by $\mathcal{H}$ on $\mathcal{D}(A)$. We denote by $[S, A]$ the bounded operator associated with the continuous extension of this form, or equivalently $-i$ times the strong derivative of the function (2.1) at $t = 0$.

A condition slightly stronger than the inclusion $S \in C^1(A)$ is provided by the following definition: $S$ belongs to $C^{1+0}(A)$, with notation $S \in C^{1+0}(A)$, if $S \in C^1(A)$ and if $[A, S]$ satisfies the Dini-type condition

$$\int_0^1 \frac{\mathrm{d}t}{t} \left\| \mathrm{e}^{-itA}[A,S] \, \mathrm{e}^{itA} - [A,S] \right\| < \infty.$$

As banachisable topological vector spaces, the sets $C^2(A)$, $C^{1+0}(A)$, $C^1(A)$ and $C^0(A) \equiv \mathscr{B}(\mathcal{H})$ satisfy the continuous inclusions [2, Sec. 5.2.4]

$$C^2(A) \subset C^{1+0}(A) \subset C^1(A) \subset C^0(A).$$

Now, let $U \in C^1(A)$ be a unitary operator with (complex) spectral measure $E^U(\,\cdot\,)$ and spectrum $\sigma(U) \subset \mathbb{S}^1 := \{z \in \mathbb{C} \mid |z| = 1\}$. If there exist a Borel set $\Theta \subset \mathbb{S}^1$, a number $a > 0$ and a compact operator $K \in \mathscr{B}(\mathcal{H})$ such that

$$E^U(\Theta)\, U^*[A,U]\, E^U(\Theta) \geq a\, E^U(\Theta) + K, \tag{2.2}$$

then one says that $U$ satisfies a Mourre estimate on $\Theta$ and that $A$ is a conjugate operator for $U$ on $\Theta$. Also, one says that $U$ satisfies a strict Mourre estimate on $\Theta$ if (2.2) holds with $K = 0$. One of the consequences of a Mourre estimate is to imply spectral properties for $U$ on $\Theta$. We recall here these spectral results in the case $U \in C^{1+0}(A)$. We also recall a result on the strong mixing property of $U$ in the case $U \in C^1(A)$ (see [7, Thm. 2.7 & Rem. 2.8] and [14, Thm. 3.1] for more details).

**Theorem 2.1** (Absolutely continuous spectrum)**.** *Let $U$ and $A$ be respectively a unitary and a self-ajoint operator in a Hilbert space $\mathcal{H}$, with $U \in C^{1+0}(A)$. Suppose there exist an open set $\Theta \subset \mathbb{S}^1$, a number $a > 0$ and a compact operator $K \in \mathscr{B}(\mathcal{H})$ such that*

$$E^U(\Theta)\,U^*[A,U]\,E^U(\Theta) \geq a\,E^U(\Theta) + K. \tag{2.3}$$

*Then, $U$ has at most finitely many eigenvalues in $\Theta$, each one of finite multiplicity, and $U$ has no singular continuous spectrum in $\Theta$. Furthermore, if (2.3) holds with $K = 0$, then $U$ has only purely absolutely continuous spectrum in $\Theta$ (no singular spectrum).*

**Theorem 2.2** (Strong mixing)**.** *Let $U$ and $A$ be respectively a unitary and a self-adjoint operator in a Hilbert space $\mathcal{H}$, with $U \in C^1(A)$. Assume that the strong limit*

$$D := \operatorname*{s-lim}_{N\to\infty} \frac{1}{N}\sum_{\ell=0}^{N-1} U^\ell\big([A,U]U^{-1}\big)U^{-\ell}$$

*exists, and suppose that $\eta(D)\mathcal{D}(A) \subset \mathcal{D}(A)$ for each $\eta \in C_c^\infty(\mathbb{R}\setminus\{0\})$. Then,*

*(a) $\lim_{N\to\infty}\langle\varphi, U^N\psi\rangle = 0$ for each $\varphi \in \ker(D)^\perp$ and $\psi \in \mathcal{H}$,*

*(b) $U|_{\ker(D)^\perp}$ has purely continuous spectrum.*

# 3 Furstenberg transformations on Cartesian products of infinite-dimensional tori

Let $\mathbb{T}^\infty \simeq \mathbb{R}^\infty/\mathbb{Z}^\infty$ be the infinite-dimensional torus with elements $x \equiv \{x_k\}_{k=1}^\infty$, and let $\widehat{\mathbb{T}^\infty}$ be the dual group of $\mathbb{T}^\infty$. For each $n \in \mathbb{N}_{\geq 1}$, let $\mu_n$ be the normalised Haar measure on $(\mathbb{T}^\infty)^n$, and let $\mathcal{H}_n := \mathsf{L}^2\big((\mathbb{T}^\infty)^n,\mu_n\big)$ be the corresponding Hilbert space. In analogy to Furstenberg transformations on Cartesian products of one-dimensional tori (see [8, Sec. 2.3]), we consider Furstenberg transformations on Cartesian products of infinite-dimensional tori $T_d : (\mathbb{T}^\infty)^d \to (\mathbb{T}^\infty)^d$, $d \in \mathbb{N}_{\geq 2}$, given by

$$T_d(x_1,x_2,\dots,x_d) := \big(x_1+\alpha, x_2+\phi_1(x_1),\dots,x_d+\phi_{d-1}(x_1,x_2,\dots,x_{d-1})\big),$$

with $\alpha \in \mathbb{T}^\infty$ such that $\{n\alpha\}_{n\in\mathbb{Z}}$ is dense in $\mathbb{T}^\infty$ and $\phi_j \in C\big((\mathbb{T}^\infty)^j;\mathbb{T}^\infty\big)$ for each $j \in \{1,\dots,d-1\}$. Since $T_d$ is invertible and preserves the measure $\mu_d$, the Koopman operator

$$W_d : \mathcal{H}_d \to \mathcal{H}_d, \quad \varphi \mapsto \varphi\circ T_d,$$

is a unitary operator. Furthermore, $W_d$ is reduced by the orthogonal decompositions

$$\mathcal{H}_d = \mathcal{H}_1 \bigoplus_{j\in\{2,\dots,d\}} \big(\mathcal{H}_j \ominus \mathcal{H}_{j-1}\big) = \mathcal{H}_1 \bigoplus_{j\in\{2,\dots,d\},\,\chi\in\widehat{\mathbb{T}^\infty}\setminus\{1\}} \mathcal{H}_{j,\chi}, \quad \mathcal{H}_{j,\chi} := \big\{\eta\otimes\chi \mid \eta\in\mathcal{H}_{j-1}\big\},$$

and the restriction $W_d|_{\mathcal{H}_{j,\chi}}$ is unitarily equivalent to the unitary operator

$$U_{j,\chi}\eta := \big(\chi\circ\phi_{j-1}\big)W_{j-1}\eta, \quad \eta\in\mathcal{H}_{j-1}.$$

In order to define later a conjugate operator for $U_{j,\chi}$, we first define a suitable group of translations on $(\mathbb{T}^\infty)^{j-1}$. For this, we choose $\{y_t\}_{t\in\mathbb{R}}$ an ergodic continuous one-parameter subgroup of $\mathbb{T}^\infty$ such that each map

$$\mathbb{R}\ni t\mapsto \big((y_t)_1,\dots,(y_t)_n\big)\in\mathbb{T}^n, \quad n\in\mathbb{N}_{\geq 1}, \tag{3.1}$$

is of class $C^1$. An example of such a subgroup $\{y_t\}_{t\in\mathbb{R}}$ is for instance given by the formula

$$(y_t)_k = y^k t \mod \mathbb{Z}, \quad t\in\mathbb{R},\ k\in\mathbb{N}_{\geq 1},$$

with $y \in \mathbb{R}$ a transcendental number (see [6, Ex. 4.1.1]). Then, we associate to $\{y_t\}_{t\in\mathbb{R}}$ the translation flow

$$F_{j-1,t}(x_1,\dots,x_{j-1}) := (x_1,\dots,x_{j-1}+y_t), \quad t \in \mathbb{R},\ (x_1,\dots,x_{j-1}) \in (\mathbb{T}^\infty)^{j-1},$$

and the translation operators $V_{j-1,t} : \mathcal{H}_{j-1} \to \mathcal{H}_{j-1}$ given by $V_{j-1,t}\eta := \eta \circ F_{j-1,t}$. Due to the continuity of the map $\mathbb{R} \ni t \mapsto y_t \in \mathbb{T}^\infty$ and of the group operation, the family $\{V_{j-1,t}\}_{t\in\mathbb{R}}$ defines a strongly continuous unitary group in $\mathcal{H}_{j-1}$ with self-adjoint generator

$$H_{j-1}\eta := \operatorname*{s-lim}_{t\to 0} it^{-1}\big(V_{j-1,t}-1\big)\eta, \quad \eta \in \mathcal{D}(H_{j-1}) := \Big\{\eta \in \mathcal{H}_{j-1} \mid \lim_{t\to 0} |t|^{-1}\big\|\big(V_{j-1,t}-1\big)\eta\big\| < \infty\Big\}.$$

When dealing with differential operators on compact manifolds, one typically does the calculations on an appropriate core of the operators such as the set of $C^\infty$-functions. But here, the are no such functions on $(\mathbb{T}^\infty)^{j-1}$, since $\mathbb{T}^\infty$ is not a manifold ($\mathbb{T}^\infty$ does not admit any differentiable structure modeled on locally convex spaces, see [4, Sec. 10.2] for details). So, we use instead the set $\mathcal{B}_{j-1}$ of Bruhat test functions on $(\mathbb{T}^\infty)^{j-1}$, whose definition is the following (see [3, Sec. 2.2] or [5] for details). Set $\mathbb{T}^0 := \{0\}$, and for each $n \in \mathbb{N}$ let $\pi_n : (\mathbb{T}^\infty)^{j-1} \to (\mathbb{T}^n)^{j-1}$ be the projection given by

$$\pi_n(x_1,\dots,x_{j-1}) := \begin{cases} (0,\dots,0) & \text{if } \ n=0 \\ \big((x_1)_1,\dots,(x_1)_n,\dots,(x_{j-1})_1,\dots,(x_{j-1})_n\big) & \text{if } \ n \ge 1. \end{cases}$$

Then,

$$\mathcal{B}_{j-1} := \bigcup_{n\in\mathbb{N}} \big\{\zeta \circ \pi_n \mid \zeta \in C^\infty\big((\mathbb{T}^n)^{j-1}\big)\big\},$$

that is, $\mathcal{B}_{j-1}$ is the set of all functions on $(\mathbb{T}^\infty)^{j-1}$ that are obtained by lifting to $(\mathbb{T}^\infty)^{j-1}$ any $C^\infty$-function on one of the Lie groups $(\mathbb{T}^n)^{j-1}$. The set $\mathcal{B}_{j-1}$ is dense in $\mathcal{H}_{j-1}$, it is left invariant by the group $\{V_{j-1,t}\}_{t\in\mathbb{R}}$, and satisfies the inclusion $\mathcal{B}_{j-1} \subset \mathcal{D}(H_{j-1})$ (to show the latter, one has to use the $C^1$-assumption (3.1)). Therefore, it follows from Nelson's theorem [1, Prop. 5.3] that $H_{j-1}$ is essentially self-adjoint on $\mathcal{B}_{j-1}$.

Now, if the (multiplication operator valued) map $\mathbb{R} \ni t \mapsto \chi \circ \phi_{j-1} \circ F_{j-1,t} \in \mathscr{B}(\mathcal{H}_{j-1})$ is strongly of class $C^1$, then $\chi \circ \phi_{j-1} \in C^1(H_{j-1})$ since

$$\chi \circ \phi_{j-1} \circ F_{j-1,t} = V_{j-1,t}\big(\chi \circ \phi_{j-1}\big)V_{j-1,-t} = \mathrm{e}^{-itH_{j-1}}\left(\chi \circ \phi_{j-1}\right)\mathrm{e}^{itH_{j-1}}.$$

Thus, the operator

$$\begin{aligned} g_{j,\chi} := \big[H_{j-1},(\chi \circ \phi_{j-1})\big](\overline{\chi} \circ \phi_{j-1}) &\equiv \text{s-}\frac{\mathrm{d}}{\mathrm{d}t}\left(i\chi \circ \phi_{j-1} \circ F_{j-1,t}\right)\left(\overline{\chi} \circ \phi_{j-1}\right)\Big|_{t=0} \\ &\equiv \text{s-}\frac{\mathrm{d}}{\mathrm{d}t}\, i\chi \circ \left(\phi_{j-1} \circ F_{j-1,t} - \phi_{j-1}\right)\Big|_{t=0} \end{aligned}$$

is a bounded self-adjoint multiplication operator in $\mathcal{H}_{j-1}$.

**Lemma 3.1.** *Take $j \in \{2,\dots,d\}$ and $\chi \in \widehat{\mathbb{T}^\infty} \setminus \{1\}$, and assume that the map $\mathbb{R} \ni t \mapsto \chi \circ \phi_{j-1} \circ F_{j-1,t} \in \mathscr{B}(\mathcal{H}_{j-1})$ is strongly of class $C^1$. Then, $U_{j,\chi} \in C^1(H_{j-1})$ with $[H_{j-1},U_{j,\chi}] = g_{j,\chi}U_{j,\chi}$.*

*Proof.* Take $\eta \in \mathscr{B}_{j-1}$. Then, the commutation of $V_{j-1,t}$ and $W_{j-1}$ and the differentiability assumption

imply that

$$
\begin{aligned}
&\big\langle H_{j-1}\eta, U_{j,\chi}\eta\big\rangle_{\mathcal{H}_{j-1}} - \big\langle \eta, U_{j,\chi}H_{j-1}\eta\big\rangle_{\mathcal{H}_{j-1}}\\
&= i\,\frac{\mathrm{d}}{\mathrm{d}t}\Big\{-\big\langle V_{j-1,t}\eta, (\chi\circ\phi_{j-1})\,W_{j-1}\eta\big\rangle_{\mathcal{H}_{j-1}} - \big\langle \eta, (\chi\circ\phi_{j-1})W_{j-1}V_{j-1,t}\eta\big\rangle_{\mathcal{H}_{j-1}}\Big\}\Big|_{t=0}\\
&= i\,\frac{\mathrm{d}}{\mathrm{d}t}\big\langle \eta, (\chi\circ\phi_{j-1}\circ F_{j-1,t} - \chi\circ\phi_{j-1})W_{j-1}V_{j-1,t}\eta\big\rangle_{\mathcal{H}_{j-1}}\Big|_{t=0}\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\big\langle \eta, \big\{\big(i\chi\circ\phi_{j-1}\circ F_{j-1,t}\big)\big(\overline{\chi}\circ\phi_{j-1}\big) - 1\big\}\,U_{j,\chi}V_{j-1,t}\eta\big\rangle_{\mathcal{H}_{j-1}}\Big|_{t=0}\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\big\langle \eta, \big\{i\chi\circ\big(\phi_{j-1}\circ F_{j-1,t} - \phi_{j-1}\big) - 1\big\}\,U_{j,\chi}V_{j-1,t}\eta\big\rangle_{\mathcal{H}_{j-1}}\Big|_{t=0}\\
&= \big\langle \eta, g_{j,\chi}\,U_{j,\chi}\eta\big\rangle_{\mathcal{H}_{j-1}},
\end{aligned}
$$

and thus the claim follows from the boundedness of $g_{j,\chi}$ and the density of $\mathscr{B}_{j-1}$ in $\mathcal{D}(\mathcal{H}_{j-1})$. $\square$

**Assumption 3.2.** *For each $j\in\{2,\dots,d\}$, the map $\phi_{j-1}\in C\big((\mathbb{T}^\infty)^{j-1};\mathbb{T}^\infty\big)$ satisfies $\phi_{j-1}=\xi_{j-1}+\eta_{j-1}$, where*

*(i) $\xi_{j-1}\in C\big((\mathbb{T}^\infty)^{j-1};\mathbb{T}^\infty\big)$ is a group homomorphism such that*

$$\mathbb{T}^\infty \ni x_{j-1}\mapsto (\chi\circ\xi_{j-1})(0,\dots,0+x_{j-1})\in\mathbb{S}^1$$

*is nontrivial for each $\chi\in\widehat{\mathbb{T}^\infty}\setminus\{1\}$,*

*(ii) $\eta_{j-1}\in C\big((\mathbb{T}^\infty)^{j-1};\mathbb{T}^\infty\big)$ is such that there exists $\widetilde{\eta}_{j-1}\in C\big((\mathbb{T}^\infty)^{j-1};\mathbb{R}^\infty\big)$ with*

$$\eta_{j-1}(x_1,\dots,x_{j-1}) = \big(\widetilde{\eta}_{j-1}(x_1,\dots,x_{j-1}) \ (\mathrm{mod}\ \mathbb{Z}^\infty)\big) \quad \textit{for each } (x_1,\dots,x_{j-1})\in(\mathbb{T}^\infty)^{j-1},$$

*and with $\mathbb{R}\ni t\mapsto \widetilde{\eta}_{j-1,k}\circ F_{j-1,t}\in\mathscr{B}(\mathcal{H}_{j-1})$ strongly of class $C^1$ for each $k\in\mathbb{N}_{\geq 1}$.*

In the next theorem, we use the fact that the map

$$\mathbb{R}\ni t\mapsto \big(\chi\circ\xi_{j-1}\big)(0,\dots,0+y_t)\in\mathbb{S}^1$$

is a character on $\mathbb{R}$, and thus of class $C^\infty$. We also use the notation

$$\xi_{j-1}^{(\chi)} := \frac{\mathrm{d}}{\mathrm{d}t}\big(\chi\circ\xi_{j-1}\big)(0,\dots,0+y_t)\Big|_{t=0}\in i\,\mathbb{R}.$$

**Theorem 3.3** (Strong mixing and unique ergodicity)**.** *Suppose that Assumption 3.2 is satisfied. Then, $W_d$ is strongly mixing in $\mathcal{H}_d\ominus\mathcal{H}_1$ and $T_d$ is uniquely ergodic with respect to $\mu_d$.*

These results of strong mixing and unique ergodicity are an extension to the case of infinite-dimensional tori of results previously obtained in the case of one-dimensional tori (see [8, Thm. 2.1] for the unique ergodicity and [10, Rem. 1] for the strong mixing property). For instance, if the functions $\eta_{j-1}$ in Assumption 3.2 were to depend only on a finite number of variables, then the strong $C^1$ regularity condition on $\eta_{j-1}$ in Assumption 3.2(ii) would reduce to a uniform Lipschitz condition of $\eta_{j-1}$ along the flow $\{F_{j-1,t}\}_{t\in\mathbb{R}}$, as in the one-dimensional case treated by Furstenberg in [8, Thm. 2.1].

*Proof.* (i) Take $j\in\{2,\dots,d\}$, $\chi\in\widehat{\mathbb{T}^\infty}\setminus\{1\}$ and $t\in\mathbb{R}$. Then, there exist $k_\chi\in\mathbb{N}_{\geq 1}$ and $n_1,\dots,n_{k_\chi}\in\mathbb{Z}$ such that

$$\chi(x_{j-1}) = \mathrm{e}^{2\pi i\sum_{k=1}^{k_\chi} n_k x_{j-1,k}},$$

with $x_{j-1} \in \mathbb{T}^\infty$ and $x_{j-1,k} \in [0,1)$ the $k$-th cyclic component of $x_{j-1}$. Therefore, we infer from Assumption 3.2 that

$$\begin{aligned}\chi \circ (\phi_{j-1} \circ F_{j-1,t} - \phi_{j-1}) &= \chi \circ (\xi_{j-1} \circ F_{j-1,t} - \xi_{j-1}) \cdot \chi \circ (\eta_{j-1} \circ F_{j-1,t} - \eta_{j-1}) \\ &= (\chi \circ \xi_{j-1})(0, \dots, 0 + y_t) \cdot \mathrm{e}^{2\pi i \sum_{k=1}^{k_\chi} n_k (\widetilde{\eta}_{j-1} \circ F_{j-1,t} - \widetilde{\eta}_{j-1})_k},\end{aligned}$$

and Lemma 3.1 implies that $U_{j,\chi} \in C^1(H_{j-1})$ with $[H_{j-1}, U_{j,\chi}] = g_{j,\chi} U_{j,\chi}$ and

$$\begin{aligned}g_{j,\chi} = \text{s-}\frac{\mathrm{d}}{\mathrm{d}t}\, i\chi \circ (\phi_{j-1} \circ F_{j-1,t} - \phi_{j-1})\Big|_{t=0} &= i\xi_{j-1}^{(\chi)} - 2\pi \sum_{k=1}^{k_\chi} n_k \left( \text{s-}\frac{\mathrm{d}}{\mathrm{d}t} \widetilde{\eta}_{j-1,k} \circ F_{j-1,t}\Big|_{t=0} \right) \\ &= i\xi_{j-1}^{(\chi)} + 2\pi i \sum_{k=1}^{k_\chi} n_k (H_{j-1}\widetilde{\eta}_{j-1,k}). \end{aligned} \tag{3.2}$$

(ii) We now proceed by induction on $d$ to prove the claims. For the case $d = 2$, we take $\chi \in \widehat{\mathbb{T}^\infty} \setminus \{1\}$ and note that point (i) implies

$$\begin{aligned}D_{2,\chi} &:= \operatorname*{s-lim}_{N\to\infty} \frac{1}{N} \sum_{\ell=0}^{N-1} (U_{2,\chi})^\ell \big([H_1, U_{2,\chi}](U_{2,\chi})^{-1}\big)(U_{2,\chi})^{-\ell} \\ &= \operatorname*{s-lim}_{N\to\infty} \frac{1}{N} \sum_{\ell=0}^{N-1} g_{2,\chi} \circ (T_1)^\ell \\ &= i\xi_1^{(\chi)} + 2\pi i \sum_{k=1}^{k_\chi} n_k \left( \operatorname*{s-lim}_{N\to\infty} \frac{1}{N} \sum_{\ell=0}^{N-1} (H_1 \widetilde{\eta}_{1,k}) \circ (T_1)^\ell \right).\end{aligned}$$

Since $T_1$ is ergodic and $H_1\widetilde{\eta}_{1,k} \in \mathsf{L}^\infty(\mathbb{T}^\infty)$, it follows by Birkhoff's pointwise ergodic theorem and Lebesgue dominated convergence theorem that

$$D_{2,\chi} = i\xi_1^{(\chi)} + 2\pi i \sum_{k=1}^{k_\chi} n_k \int_{\mathbb{T}^\infty} \mathrm{d}\mu_1 \, (H_1\widetilde{\eta}_{1,k}) = i\xi_1^{(\chi)} + 2\pi i \sum_{k=1}^{k_\chi} n_k \int_{\mathbb{T}^\infty} \mathrm{d}\mu_1 \, \langle 1, H_1\widetilde{\eta}_{1,k} \rangle_{\mathcal{H}_1} = i\xi_1^{(\chi)}.$$

Now, since the character $\mathbb{T}^\infty \ni x_{j-1} \mapsto (\chi \circ \xi_1)(0, \dots, 0 + x_{j-1}) \in \mathbb{S}^1$ is nontrivial and the subgroup $\{y_t\}_{t\in\mathbb{R}}$ ergodic, we have $\xi_1^{(\chi)} \neq 0$ (see [6, Thm. 4.1.1']). Thus, $D_{2,\chi} \neq 0$, and we deduce from Theorem 2.2(a) that $U_{2,\chi}$ is strongly mixing. Since this is true for each $\chi \in \widehat{\mathbb{T}^\infty} \setminus \{1\}$, and since $U_{2,\chi}$ is unitarily equivalent to $W_2|_{\mathcal{H}_{2,\chi}}$, we infer that $W_2$ is strongly mixing in $\oplus_{\chi\in\widehat{\mathbb{T}^\infty}\setminus\{1\}} \mathcal{H}_{2,\chi} = \mathcal{H}_2 \ominus \mathcal{H}_1$.

To show that $T_2$ is uniquely ergodic with respect to $\mu_2$, we take an eigenvector of $W_2$ with eigenvalue 1, that is, a vector $\varphi \in \mathcal{H}_2$ such that $W_2\varphi = \varphi$. Since $W_2$ is strongly mixing in $\mathcal{H}_2 \ominus \mathcal{H}_1$, $W_2$ has purely continuous spectrum in $\mathcal{H}_2 \ominus \mathcal{H}_1$ (see Theorem 2.2(b)), and thus $\varphi = \eta \otimes 1$ for some $\eta \in \mathcal{H}_1$. So,

$$W_2\varphi = \varphi \iff W_2(\eta \otimes 1) = \eta \otimes 1 \iff W_1\eta = \eta,$$

and thus $\eta$ is an eigenvector of $W_1$ with eigenvalue 1. It follows that $\eta$ is constant $\mu_1$-almost everywhere due to the ergodicity of $T_1$. Therefore, $\varphi$ is constant $\mu_2$-almost everywhere, and $T_2$ is ergodic. This implies that $T_2$ is uniquely ergodic because ergodicity implies unique ergodicity for transformations such as $T_2$ (see [6, Thm. 4.2.1]).

Now, assume the claims are true for $d - 1 \geq 1$. Then, $W_{d-1}$ is strongly mixing in $\mathcal{H}_{d-1} \ominus \mathcal{H}_1$. Furthermore, a calculation as in the case $d = 2$ shows that $W_d$ is strongly mixing in $\oplus_{\chi\in\widehat{\mathbb{T}^\infty}\setminus\{1\}} \mathcal{H}_{d,\chi} = \mathcal{H}_d \ominus \mathcal{H}_{d-1}$. This implies that $W_d$ is strongly mixing in

$$(\mathcal{H}_{d-1} \ominus \mathcal{H}_1) \oplus (\mathcal{H}_d \ominus \mathcal{H}_{d-1}) = \mathcal{H}_d \ominus \mathcal{H}_1.$$

This, together with the unique ergodicity of $T_{d-1}$, allows us to show that $T_d$ is uniquely ergodic as in the case $d = 2$. □

We know from the proof of Theorem 3.3 that if Assumption 3.2 is satisfied, then $U_{j,\chi} \in C^1(H_{j-1})$ with $[H_{j-1}, U_{j,\chi}] = g_{j,\chi} U_{j,\chi}$. So, it follows from [7, Sec. 4] that the operator

$$A^{(N)}_{j,\chi}\eta := \frac{1}{N}\sum_{\ell=0}^{N-1}(U_{j,\chi})^{\ell} H_{j-1}(U_{j,\chi})^{-\ell}\eta, \quad N \in \mathbb{N}_{\geq 1},\ \eta \in \mathcal{D}\big(A^{(N)}_{j,\chi}\big) := \mathcal{D}(H_{j-1}),$$

is self-adjoint, and that $U_{j,\chi} \in C^1\big(A^{(N)}_{j,\chi}\big)$ with

$$\big[A^{(N)}_{j,\chi}, U_{j,\chi}\big] = g^{(N)}_{j,\chi} U_{j,\chi} \qquad \text{and} \qquad g^{(N)}_{j,\chi} := \frac{1}{N}\sum_{\ell=0}^{N-1} g_{j,\chi} \circ (T_{j-1})^{\ell}.$$

**Theorem 3.4** (Countable Lebesgue spectrum)**.** *Suppose that Assumption 3.2 is satisfied, and assume for each $j \in \{2, \dots, d\}$ and $k \in \mathbb{N}_{\geq 1}$ that $H_{j-1}\widetilde{\eta}_{j-1,k} \in C\big((\mathbb{T}^\infty)^{j-1}\big)$ and that*

$$\int_0^1 \frac{\mathrm{d}t}{t}\,\big\|(H_{j-1}\widetilde{\eta}_{j-1,k}) \circ F_{j-1,t} - (H_{j-1}\widetilde{\eta}_{j-1,k})\big\|_{\mathsf{L}^\infty((\mathbb{T}^\infty)^{j-1})} < \infty. \tag{3.3}$$

*Then, $W_d$ has countable Lebesgue spectrum in $\mathcal{H}_d \ominus \mathcal{H}_1$.*

*Proof.* Take $j \in \{2, \dots, d\}$ and $\chi \in \widehat{\mathbb{T}^\infty} \setminus \{1\}$. Then, we know from Lemma 3.1 that $U_{j,\chi} \in C^1(H_{j-1})$ with $[H_{j-1}, U_{j,\chi}] = g_{j,\chi} U_{j,\chi}$. Furthermore, (3.2) and (3.3) imply that

$$\begin{aligned}
&\int_0^1 \frac{\mathrm{d}t}{t}\,\big\|\,\mathrm{e}^{-itH_{j-1}}\, g_{j,\chi}\, \mathrm{e}^{itH_{j-1}} - g_{j,\chi}\big\|_{\mathscr{B}(\mathcal{H}_{j-1})}\\
&= \int_0^1 \frac{\mathrm{d}t}{t}\,\big\|g_{j,\chi} \circ F_{j-1,t} - g_{j,\chi}\big\|_{\mathsf{L}^\infty((\mathbb{T}^\infty)^{j-1})}\\
&\leq 2\pi \sum_{k=1}^{k_\chi} |n_k| \int_0^1 \frac{\mathrm{d}t}{t}\,\big\|(H_{j-1}\widetilde{\eta}_{j-1,k}) \circ F_{j-1,t} - (H_{j-1}\widetilde{\eta}_{j-1,k})\big\|_{\mathsf{L}^\infty((\mathbb{T}^\infty)^{j-1})}\\
&< \infty.
\end{aligned}$$

So, we obtain that $U_{j,\chi} \in C^{1+0}(H_{j-1})$ with $[H_{j-1}, U_{j,\chi}] = g_{j,\chi} U_{j,\chi}$, and thus deduce from [7, Sec. 4] that $U_{j,\chi} \in C^{1+0}\big(A^{(N)}_{j,\chi}\big)$ with $\big[A^{(N)}_{j,\chi}, U_{j,\chi}\big] = g^{(N)}_{j,\chi} U_{j,\chi}$. Now, $H_{j-1}\widetilde{\eta}_{j-1,k} \in C\big((\mathbb{T}^\infty)^{j-1}\big)$ and $T_{j-1}$ is uniquely ergodic with respect to $\mu_{j-1}$ due to Theorem 3.3. Thus, we infer from (3.2) that

$$\begin{aligned}
\lim_{N\to\infty} g^{(N)}_{j,\chi} &= i\,\xi^{(\chi)}_{j-1} + 2\pi i \sum_{k=1}^{k_\chi} n_k \,\frac{1}{N}\sum_{\ell=0}^{N-1} (H_{j-1}\widetilde{\eta}_{j-1,k}) \circ (T_{j-1})^{\ell}\\
&= i\,\xi^{(\chi)}_{j-1} + 2\pi i \sum_{k=1}^{k_\chi} n_k \int_{(\mathbb{T}^\infty)^{j-1}} \mathrm{d}\mu_{j-1}\,\big\langle 1, H_{j-1}\widetilde{\eta}_{j-1,k}\big\rangle_{\mathcal{H}_{j-1}}\\
&= i\,\xi^{(\chi)}_{j-1}
\end{aligned}$$

uniformly on $(\mathbb{T}^\infty)^{j-1}$. Since $\xi^{(\chi)}_{j-1} \neq 0$ by the proof of Theorem 3.3, one has $\big|g^{(N)}_{j,\chi}\big| > 0$ if $N$ is large enough. So, $\big|g^{(N)}_{j,\chi}\big| \geq a$ with $a := \inf_{x \in (\mathbb{T}^\infty)^{j-1}} \big|g^{(N)}_{j,\chi}(x)\big| > 0$, and $U_{j,\chi}$ satisfies the following strict Mourre estimate on $\mathbb{S}^1$:

$$(U_{j,\chi})^*\big[\mathrm{sgn}\big(g^{(N)}_{j,\chi}\big)A^{(N)}_{j,\chi}, U_{j,\chi}\big] = (U_{j,\chi})^*\big|g^{(N)}_{j,\chi}\big|\,U_{j,\chi} \geq a.$$

Therefore, all the assumptions of Theorem 2.1 are satisfied, and thus $U_{j,\chi}$ has purely absolutely continuous spectrum. Since this occurs for each $j \in \{2, \dots, d\}$ and $\chi \in \widehat{\mathbb{T}^\infty} \setminus \{1\}$, and since $U_{j,\chi}$ is unitarily equivalent to $W_d|_{\mathcal{H}_{j,\chi}}$, one infers that $W_d$ has purely absolutely continuous spectrum in $\bigoplus_{j \in \{2,\dots,d\},\, \chi \in \widehat{\mathbb{T}^\infty}\setminus\{1\}} \mathcal{H}_{j,\chi} = \mathcal{H}_d \ominus \mathcal{H}_1$. Finally, the fact that $W_d$ has has countable Lebesgue spectrum in $\mathcal{H}_d \ominus \mathcal{H}_1$ can be shown in a similar way as in [10, Lemmas 3 & 4]. $\square$

The result of Theorem 3.4 is an extension to the case of infinite-dimensional tori of results previously obtained in the case of one-dimensional tori (see [10, Cor. 3] or [15, Thm. 5.3]).

## References


[1] W. O. Amrein. *Hilbert space methods in quantum mechanics*. Fundamental Sciences. EPFL Press, Lausanne, 2009.

[2] W. O. Amrein, A. Boutet de Monvel, and V. Georgescu. *$C_0$-groups, commutator methods and spectral theory of $N$-body Hamiltonians*, volume 135 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1996.

[3] A. Bendikov and L. Saloff-Coste. On the sample paths of Brownian motions on compact infinite dimensional groups. *Ann. Probab.*, 31(3):1464–1493, 2003.

[4] V. I. Bogachev. Differentiable measures and the Malliavin calculus. *J. Math. Sci. (New York)*, 87(4):3577–3731, 1997. Analysis, 9.

[5] F. Bruhat. Distributions sur un groupe localement compact et applications à l'étude des représentations des groupes $\wp$-adiques. *Bull. Soc. Math. France*, 89:43–75, 1961.

[6] I. P. Cornfeld, S. V. Fomin, and Y. G. Sinaĭ. *Ergodic theory*, volume 245 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York, 1982. Translated from the Russian by A. B. Sosinskiĭ.

[7] C. Fernández, S. Richard, and R. Tiedra de Aldecoa. Commutator methods for unitary operators. *J. Spectr. Theory*, 3(3):271–292, 2013.

[8] H. Furstenberg. Strict ergodicity and transformation of the torus. *Amer. J. Math.*, 83:573–601, 1961.

[9] G. Greschonig. Nilpotent extensions of Furstenberg transformations. *Israel J. Math.*, 183:381–397, 2011.

[10] A. Iwanik, M. Lemańczyk, and D. Rudolph. Absolutely continuous cocycles over irrational rotations. *Israel J. Math.*, 83(1-2):73–95, 1993.

[11] R. Ji. *ON THE CROSSED PRODUCT C*-ALGEBRAS ASSOCIATED WITH FURSTENBERG TRANSFORMATIONS ON TORI*. ProQuest LLC, Ann Arbor, MI, 1986. Thesis (Ph.D.)–State University of New York at Stony Brook.

[12] H. Osaka and N. C. Phillips. Furstenberg transformations on irrational rotation algebras. *Ergodic Theory Dynam. Systems*, 26(5):1623–1651, 2006.

[13] K. Reihani. $K$-theory of Furstenberg transformation group $C^*$-algebras. *Canad. J. Math.*, 65(6):1287–1319, 2013.

[14] R. Tiedra de Aldecoa. Commutator criteria for strong mixing. to appear in Ergodic Theory Dynam. Systems, preprint on `http://arxiv.org/abs/1406.5777`.

[15] R. Tiedra de Aldecoa. Commutator methods for the spectral analysis of uniquely ergodic dynamical systems. *Ergodic Theory and Dynamical Systems*, FirstView:1–24, 10 2014.